Heun Method

The simplest example of a predictor corrector method

A “marching” method for obtaining ordered pairs starting with an initial value set
Heun Example

The application of the Heun method using a known form for the differential equation,

Notation Examples:

(1) $f(z, F(z)) = z^3 + 4z^2 + 12z - 75$ This function could be labeled $f(z)$ since there was no $F(z)$ term in the expression.

\[ f(z) = z^3 + 4z^2 + 12z - 75 \]

(2)
Heun Example

The application of the Heun method using a known form for the differential equation,

Notation Examples:

(1) \[ f(z, F(z)) = z^3 + 4z^2 + 12z - 75 \]  
   This function could be labeled \( f(z) \) since there was no \( F(z) \) term in the expression.  
   \[ f(z) = z^3 + 4z^2 + 12z - 75 \]

(2) \[ f(x, y) = 4e^{0.8x} - 0.5 F(x) \]  
   This function cannot be labeled \( f(z) \) since there is a \( F(z) \) term in the expression.

Additional Required Information

initial ordered pair, \((0,2)\)
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

\[ F(0) = 2 \]

\[ F(1) = 2 + (\Delta x) \frac{dy}{dx} \bigg|_{x=0} = 2 + (\Delta x) f(x,F(x)) \bigg|_{x=0} \]

\[ F(2) = F(1) + (\Delta x) \frac{dy}{dx} \bigg|_{x=0 + \Delta x} = 2 + (\Delta x) f(x,F(x)) \bigg|_{x=0 + \Delta x} \]

\[ F(2) = F(1) + (\Delta x) \frac{dy}{dx} \bigg|_{x=0 + 2\Delta x} = 2 + (\Delta x) f(x,F(x)) \bigg|_{x=0 + 2\Delta x} \]

The objective is to obtain the other value in the ordered pair \((1,?)\). After that value is obtained at an approximation level that satisfies the uncertainty of measurement system you “march” on to obtain the two values in the next ordered pair in the sequence.

For predictor/corrector marching methods it is a two step process for each corrected “answer”
Two correction Heun calculation

Example:  \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

Two step process for each corrected “answer”

Step #1: Use the Euler to get a prediction of the “answer”.

solution using Euler \( F(x) = F(x_0) + [f(x_0, F(x_0)](h) \)

(1, ?) current value of F(x)

(0, 2) Initial ordered pair

(1, ?) desired ordered pair
Two correction Heun calculation

Example: \( f(x, F(x)) = 4e^{0.8x} - 0.5 F(x) \)

Two step process for each corrected “answer”

Step #1: Use the Euler to get a prediction of the “answer”.

solution value using Euler

\[ F(x) = F(x_0) + [f(x_0, F(x_0))](h) \]

current value of \( F(x) \)

\( (1, ?) \)

\[ F(1) = 2 + [f(0,2)](1) = ? \]

\[ [f(0,2)] = 4e^0 - 0.5(2) = 3 \]
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 \, e^{0.8x} - 0.5 \, F(x) \)

Two step process for each corrected “answer”

Step #1: Use the Euler to get a prediction of the “answer”.

\[ F(x) = F(x_0) + [f(x_0, F(x_0))](h) \]

current value of \( F(x) \)

\( (1, ?) \)

\[ F(1) = 2 + [f(0,2)](1) = ? \]

\[ [f(0,2)] = 4 \, e^0 - 0.5 \, (2) = 3 \]

value for \( F(1) \) obtained using Euler (the predicted value)

\[ F(1)^0 = 2 + 3(1) = 5 \]

(A predicted solution to differential equation based on initial Euler method results.)

When \( x = 1 \) the \( F(x) \) value is suppose to equal 5.
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 \, e^{0.8x} - 0.5 \, F(x) \)

Two step process for each corrected “answer”

Step #1: Use the Euler to get a prediction of the “answer”.

solution value using Euler

\[
F(x) = F(x_0) + [ f(x_0, F(x_0)](h)
\]

\[(1, ?)\]

\[F(1) = 2 + [ f(0,2)](1) = ?\]

\[ [ f(0,2)] = 4 \, e^0 - 0.5 \, (2) = 3 \]

value for \( F(1) \) obtained using Euler (the predicted value)

\[ F^0(1) = 2 + 3(1) = 5 \]

(A predicted solution to differential equation based on initial Euler method results.)

When \( x = 1 \) the \( F(x) \) value is suppose to equal 5.

If this predicted value isn’t good enough, it is time to move on to the correction step!
Example: \( f(x, F(x)) = 4e^{0.8x} - 0.5F(x) \)

Two correction Heun calculation

Two step process for each corrected “answer”

**Step #1:** Use the Euler to get a prediction of the “answer”.

- Solution value using Euler:
  \[ F(x) = F(x_0) + \left[ f(x_0, F(x_0)) \right](h) \]

- Initial ordered pair: \((0, 2)\)
- Desired ordered pair: \((1, ?)\)
- Predicted ordered pair: \((1, 5)\)

(A predicted solution to differential equation when \(x = 1\))

- \(F(1) = 2 + \left[ f(0, 2) \right](1) = 5\)

- \(f(0, 2) = 4e^{0} - 0.5(2) = 3\)

- \([f(0, 2)] = 4e^0 - 0.5(2) = 3\)

**Step #2:** the correction to the Euler prediction of the “answer”.

- \((1, 5)\)

- \(f(x, F(x)) = 4e^{0.8x} - 0.5F(x)\)

- \([f_p(1, 5)] = 4e^{0.8(1)} - (0.5)F(1) = ?\)

- \([f_p(1, 5)] = 4e^{0.8(1)} - (0.5)(5) = 6.402\)
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

Solution value using Euler:

\[
F(x) = F(x_0) + [f(x_0, F(x_0))](h)
\]

First correction

\[
F_c^1(x) = \frac{1}{2} [f_p(1, 5)] = 4 e^{0.8(1)} - (0.5) F(1) = \text{?}
\]

\[
[f_p(1, 5)] = 4 e^{0.8(1)} - (0.5) 5 = 6.402
\]

Step #2: the correction to the Euler prediction of the “answer”.

\[
1, 5
\]

Initial ordered pair \( (0, 2) \)

Desired ordered pair \( (1, ?) \)

Predicted ordered pair \( (1, 5) \)

First correction \( (1, ?) \)
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

Solution value using Euler:
\[
F(x) = F(x_0) + [f(x_0, F(x_0))](h)
\]

Current value of \( F(x) \)

Step #2: the correction to the Euler prediction of the “answer”.

\[
(1, 5)
\]

Heun first correction to solution value, \( F(1) \).

\[
\begin{align*}
[f_p(1, 5)] &= 4 e^{0.8(1)} - (0.5) F(1) = ? \\
\end{align*}
\]

\[
\begin{align*}
[f_p(1, 5)] &= 4 e^{0.8(1)} - (0.5) 5 = 6.402
\end{align*}
\]
Two correction Heun calculation

Example: \( f(x, F(x)) = 4e^{0.8x} - 0.5F(x) \)

Solution value using Euler:
\[
F(x) = F(x_0) + \left[ f(x_0), F(x_0) \right](h)
\]

Desired ordered pair
\[
(1, \ ?)
\]

Predicted ordered pair
\[
(1, 5)
\]

Initial ordered pair
\[
(0, 2)
\]

Step #2: the correction to the Euler prediction of the “answer”.

\[
\begin{align*}
(1, 5) \\
\left[ f_p(1, 5) \right] &= 4e^{0.8(1)} - (0.5) F(1) = ? \\
\left[ f_p(1, 5) \right] &= 4e^{0.8(1)} - (0.5) 5 = 6.402
\end{align*}
\]

Heun first correction to solution value, \( F(1) \).

\[
F^1_c(x) = 2 + \left\{ \frac{[4e^{0.8(0)} - 0.5(2)] + [4e^{0.8(1)} - 0.5(5)]}{2} \right\} (1) = ?
\]
Two correction Heun calculation

Example: \( f(x, F(x)) = 4e^{0.8x} - 0.5 F(x) \)

solution value using Euler

\[
F(x) = F(x_0) + f(x_0, F(x_0)) \cdot (h)
\]

current value of \( F(x) \)

**Step #2: the correction to the Euler prediction.**

\[
(1, 5)
\]

\[
[f_p(1, 5)] = 4e^{0.8(1)} - (0.5) F(1) = 6.402
\]

Heun first correction to solution value, \( F(1) \).

\[
F^1_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\} (1) = 2 + \{ \}
\]

current value of \( F(x) \)
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

solution value using Euler \( F(x) = F(x_0) + [ f(x_0, F(x_0))]h \)

current value of \( F(x) \)

Step #2: the correction to the Euler prediction.

\( (1, 5) \)

\[ [f_p(1, 5)] = 4 e^{0.8(1)} - (0.5) F(1) \]

\[ f_p(1, 5) = 4 e^{0.8(1)} - (0.5) 5 = 6.402 \]

Heun first correction to solution value, \( F(1) \).

\[ F_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\} (1) = 2 + \{4.702\} (1) = 6.702 \rightarrow f(1, 6.702) = [4 e^{0.8(1)} - 0.5 (6.702)] \]

\( (1, 6.702) \)

When \( x = 1 \) the \( F(x) \) value is now suppose to equal 6.702.
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 \, e^{0.8x} - 0.5 \, F(x) \)

solution value

\[ F(x) = F(x_0) + \left[ f(x_0, F(x_0)) \right](h) \]

current value of \( F(x) \)

Step #2: the correction to the Euler prediction.

\[ (1, 5) \]

\[ [f_p(1, 5)] = 4 \, e^{0.8(1)} - (0.5) \, F(1) = ? \]

\[ [f_p(1, 5)] = 4 \, e^{0.8(1)} - (0.5) \, 5 = 6.402 \]

Heun first correction to solution value, \( F(1) \).

\[ F_c^1(x) = 2 + \left\{ \left[ \frac{3 + 6.402}{2} \right] \right\}(1) = 2 + \{4.702\} \, (1) = 6.702 \rightarrow f(1, 6.702) = [4 \, e^{0.8(1)} - 0.5 \, (6.702)] \]

\[ (1, 6.702) \]

When \( x = 1 \) the \( F(x) \) value is now suppose to equal \( 6.702 \).

If this corrected value still isn’t good enough, it is time to move on to a second correction step!!
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

solution value using Euler \( F(x) = F(x_0) + [ f(x_0, F(x_0))](h) \)

current value of \( F(x) \)

Original predicted value of \( F(x) = 5 \) when \( x = 1 \)

\[(1, 5)\]

Heun first correction to solution value, \( F(1) \).

\[
F^1_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\}(1) = 2 + \{4.702\} (1) = 6.702 \rightarrow f(1, 6.702) = [4 e^{0.8(1)} - 0.5 (6.702)]
\]

\[(1, 6.702)\]

Heun second correction to the predicted value of \( F(1) \)

\[
F^2_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\} (1)
\]

continue to use the current value of \( F(x) \)

\[
F^2_c(x) = 2 +
\]

\[(0, 2) \quad \text{Initial ordered pair}
(1, ?) \quad \text{desired ordered pair}
(1, 5) \quad \text{predicted ordered pair}
(1, 6.702) \quad \text{First correction}
(1, ?) \quad \text{Second correction}
\]
Two correction Heun calculation

Example: \( f(x, F(x)) = 4 e^{0.8x} - 0.5 F(x) \)

solution value using Euler
\[ F(x) = F(x_0) + \left[ f(x_0, F(x_0)) \right](h) \]
current value of \( F(x) \)

Original predicted value of \( F(x) \) = 5 when \( x = 1 \)

(1, 5)

Heun first correction to solution value, \( F(1) \).

\[
F^1_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\}(1) = 2 + \{4.702\}(1) = 6.702 \rightarrow f(1, 6.702) = [4 e^{0.8(1)} - 0.5(6.702)]
\]

(1, 6.702)

Heun second correction to the predicted value of \( F(1) \)

\[
F^2_c(x) = 2 + \left\{ \frac{[4 e^{0.8(0)} - 0.5(2)] + [4 e^{0.8(1)} - 0.5(6.702)]}{2} \right\}(1) = ?
\]
Two correction Heun calculation

Example: \( f(x, F(x)) = 4e^{0.8x} - 0.5F(x) \)

desired ordered pair \((1, 6.382)\)

predicted ordered pair \((1, 5)\)

Initial ordered pair \((0, 2)\)

Original predicted value of \(F(x) = 5\) when \(x = 1\)

Heun first correction to solution value, \(F(1)\).

\[
F_c^1(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\} (1) = 2 + \{4.702\} (1) = 6.702 \rightarrow f(1, 6.702) = [4e^{0.8(1)} - 0.5(6.702)]
\]

\((1, 6.702)\)

Heun second correction to the predicted value of \(F(1)\)

\[
F_c^2(x) = 2 + \left\{ \frac{[4e^{0.8(0)} - 0.5(2)] + [4e^{0.8(1)} - 0.5(6.702)]}{2} \right\} (1) = 6.382
\]

\((1, 6.382)\)
Two correction Heun calculation

Example: $f(x, F(x)) = 4e^{0.8x} - 0.5F(x)$

solution value using Euler $F(x) = F(x_0) + \left[ f(x_0, F(x_0)) \right](h)$

current value of $F(x)$

Original predicted value of $F(x) = 5$ when $x = 1$

$(1, 5)$

Heun first correction to solution value, $F(1)$.

$F^1_c(x) = 2 + \left\{ \frac{3 + 6.402}{2} \right\}(1) = 2 + \{4.702\}(1) = 6.702 \rightarrow f(1, 6.702) = [4e^{0.8(1)} - 0.5(6.702)]$

$(1, 6.702)$

Heun second correction to the predicted value of $F(1)$

$F^2_c(x) = 2 + \left\{ \frac{[4e^{0.8(0)} - 0.5(2)] + [4e^{0.8(1)} - 0.5(6.702)]}{2} \right\}(1) = 6.382$

$(1, 6.382)$

When $x = 1$ the $F(x)$ value is now really suppose to equal 6.382

$(0, 2)$ Initial ordered pair

$(1, 6382)$ desired ordered pair
Two correction Heun calculation

\[ f(x, F(x) ) = 4 e^{0.8x} - 0.5 F(x) \]

The analytical solution of this example differential equation gives the correct ordered pair as \((1, 6.195)\).

Write a MATLAB routine that will do this two correction numerical method.

- Have the routine check the results provided to the left.
- Have the routine also calculate and plot the relative errors as well.

Initial ordered pair: \((0, 2)\)
Correct ordered pair: \((1, 6.195)\)
Predicted ordered pair: \((1, 5)\)
1st correction: \((1, 6.702)\)
2nd correction: \((1, 6.382)\)
Two correction Heun calculation

End of Presentation